Complex-analytic approach to quantum groups

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Quantum group example (not very popular but interesting): a deformation of $U(\mathfrak{af}_1)$, the universal enveloping algebra of the non-abelian 2-dimensional complex Lie algebra (corresponding to the group of affine transformations of $\mathbb{C}$) .
Generators X and Y subject to $[X, Y] = Y$.
Deformed multiplication ($\hbar$ is a hint of Planck constant)

$$[X, Y] = \frac{\sinh \hbar Y}{\sinh \hbar} .$$

(Aizawa–Sato, 1994, similar relations in Majid, 1995)
Deformed comultiplication and antipode:

$$\Delta\colon X \mapsto X \otimes e^{-\hbar Y} + e^{\hbar Y} \otimes X, \qquad Y \mapsto 1 \otimes Y + Y \otimes 1;$$
$$S\colon X \mapsto -X - \frac{\hbar}{\sinh \hbar} \sinh \hbar Y, \qquad Y \mapsto -Y.$$

Common problem: sinh and other functions are not polynomials.

Common solution: $\hbar$-adic form: treat $\hbar$ as a formal variable and $U_\hbar(\mathfrak{af}_1)$ as an algebra over $\mathbb{C}[[\hbar]]$ (formal power series).

Another solution: sinh etc are entire functions! Where we can take the entire function of an element? In a Banach algebra. Moreover, entire functions can be applied to elements of Arens-Michael algebras (projective limits of Banach als.). Now we treat $\hbar$ as a fixed number in $\mathbb{C}$ (with $\sinh\hbar \neq 0$).

Let $\widetilde{U}(\mathfrak{af}_1)_\hbar$ be the universal Arens-Michael algebra generated by X and Y subject to $[X, Y] = \sinh\hbar Y/(\sinh\hbar)$, i.e. the quotient of the algebra $\mathscr{F}_n$ of free entire functions over the closed two-sided ideal generated by the relation.

Free entire functions:

$$\mathscr{F}_n = \Big\{a = \sum_{\alpha\in W_n} c_\alpha \zeta_\alpha \colon \; \|a\|_\rho := \sum_\alpha |c_\alpha|\, \rho^{|\alpha|} < \infty \;\forall \rho > 0\Big\}.$$

$\zeta_1, \ldots, \zeta_n$ are the generators. In our case, $\zeta_1 \mapsto X$, $\zeta_2 \mapsto Y$.

So $\widetilde{U}(\mathfrak{af}_1)_\hbar$ is holomorphically finitely generated (in short, HFG) in the sense of Pirkovskii, 2014.

Theorem (A.,2020)

As a LCS,

$$\widetilde{U}(\mathfrak{af}_1)_\hbar \cong \prod_{n\in\mathbb{Z}} \mathbb{C}[[Y_n]] \mathbin{\widehat{\otimes}} \mathcal{O}(\mathbb{C}),$$

where (Y_n) are formal variables corresponding to zeros of $\sinh$.

Here $\widehat{\otimes}$ is the complete projective tensor product of locally convex spaces and $\mathcal{O}(\mathbb{C})$ is the algebra of holomorphic functions.
($\widehat{\otimes}$ is compatible with holomorphic functions:
$\mathcal{O}(U) \mathbin{\widehat{\otimes}} \mathcal{O}(V) \cong \mathcal{O}(U \times V)$.)

Hopf algebra structure?

A Hopf $\widehat{\otimes}$-algebra (read as "topological Hopf algebra") = Hopf algebra in the monoidal category of complete locally convex spaces with $\widehat{\otimes}$.

Theorem (A.,2020)

$\widetilde{U}(\mathfrak{af}_1)_\hbar$ is a Hopf $\widehat{\otimes}$-algebra with respect to Δ, ε and S determined by the formulas above.

Definition

Hopf HFG algebra:= Hopf $\widehat{\otimes}$-algebra that is HFG.

Thus, $\widetilde{U}(\mathfrak{af}_1)_\hbar$ is a Hopf HFG algebra.

Drinfeld-Jimbo algebras.

DJ algebras are deformations of $U(\mathfrak{g})$ (the universal enveloping algebra), where $\mathfrak{g}$ is a semisimple Lie algebra. (All algebras over $\mathbb{C}$.)
$U(\mathfrak{g})$ is the universal associative algebra generated by $\mathfrak{g}$.
E.g., $\mathfrak{sl}_2$ is the Lie algebra with generators E, F, H subject to

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

and $U(\mathfrak{sl}_2)$ is the un. associative algebra subject to this relations.

$\hbar$-adic Drinfeld-Jimbo algebra of $\mathfrak{sl}_2$

Replace the 3d relation by

$$[E, F] = \frac{\sinh \hbar H}{\sinh \hbar}.$$

$\hbar$ is a letter not a number and $U_\hbar(\mathfrak{sl}_2)$ is an algebra over formal series in $\hbar$.

Two ways to specialize $\hbar$ to $\mathbb{C}$. (Below $K := e^H$ and $q := e^\hbar$.)

1st: traditional, algebraic exponential form

$U_q(\mathfrak{sl}_2)$ is the universal associative algebra subject to

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$q \in \mathbb{C}$ and $q \neq \pm 1$.

2nd, new, complex-analytic form

Let $\hbar \in \mathbb{C}$ and $\sinh \hbar \neq 0$. Then $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ is defined as the universal Arens-Michael algebra generated by E, F, H subject to

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{\sinh \hbar H}{\sinh \hbar}.$$

Proposition (A., 2020)

$\widetilde{U}(\mathfrak{sl}_2)_\hbar$ is an infinite-dimensional Hopf HFG algebra w.r.t. the comultiplication Δ, counit ε and antipode S determined by same formulas as for the $\hbar$-adic form $U_\hbar(\mathfrak{sl}_2)$.

Explicit formulas:

$$\Delta\colon H \mapsto H \otimes 1 + 1 \otimes H,$$

$$\Delta\colon E \mapsto E \otimes e^{\hbar H} + 1 \otimes E,$$

$$\Delta\colon F \mapsto F \otimes 1 + e^{-\hbar H} \otimes F;$$

$$S\colon H \mapsto -H, \quad E \mapsto -Ee^{-\hbar H}, \quad F \mapsto -e^{\hbar H} F;$$

$$\varepsilon\colon H, E, F \mapsto 0.$$

My initial motivation: to find the Arens-Michael envelope of $U_q(\mathfrak{sl}_2)$.
The AM envelope arises when we approximate an algebra by Banach algebras.

If A is an associative $\mathbb{C}$-algebra, then the AM envelope is the completion $\widehat{A}$ w.r.t. all possible submultiplicative seminorms ($\|ab\| \leqslant \|a\|\,\|b\|$) with the homomorphism $\iota\colon A \to \widehat{A}$ (eq. the projective limit of all Banach algebra completions).

Universal property

Every homomorphism from A to a Banach algebra factors on ι.

The structure of $\widehat{A}$ is closely related to representations of A on Banach spaces.

The AM envelope functor can be considered as a functor from NC algebraic geometry to NC complex-analytic geometry (because it works well in the commutative case, Pirkovskii, 2006)

H is a Hopf algebra $\Rightarrow \widehat{H}$ is a Hopf $\widehat{\otimes}$-algebra (Pirkovskii).
So: H is an affine (f.g.) Hopf alg. $\Rightarrow \widehat{H}$ is an Hopf HFG alg.

$\widehat{U}_q(\mathfrak{sl}_2)$ and $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ are HFG. Structure of underlying LCSs?

AM envelope of the algebraic form

The case $|q| = 1$ — only partial results.

Pedchenko (2015) found a power series description of $\widehat{U}_q(\mathfrak{sl}_2)$ for $|q| = 1$ (using analytical Ore extensions).

Proposition (A., 2020)

$q \in \mathbb{C}$ is not a root of unity, $|q| = 1 \Rightarrow \exists$ topologically irreducible representations of $U_q(\mathfrak{sl}_2)$ on a Hilbert space.

The argument does not work when q is a root of unity.

Well known

If q is a root of unity, then all irreducible representations of $U_q(\mathfrak{sl}_2)$ are finite dimensional (three continuous parameters). But there are f.d. representations that are not completely reducible.

When $|q| \neq 1$, the structure of $\widehat{U}_q(\mathfrak{sl}_2)$ is similar to that in the classical undeformed case.

Theorem (Joseph Taylor, 1972, undeformed case)

$\mathfrak{g}$ is a semisimple Lie algebra $\Rightarrow \widehat{U}(\mathfrak{g})$ is topologically isomorphic to the direct product of a countable family of full matrix algebras.

Idea of Taylor's proof: pass from $\mathfrak{g}$ to a semisimple Lie group, which has a compact Lie group as a real form. Next use the representation theory of compact groups.

There is no similar technical tool for $U_q(\mathfrak{sl}_2)$.

Idea of an alternative proof: use an algebraic argument to show that E and F are nilpotent in a Banach algebra.

Theorem (A., 2020)

Let $|q| \neq 1$.
(A) The range of any homomorphism from $U_q(\mathfrak{sl}_2)$ to a Banach algebra is finite dimensional.
(B) The AM envelope $\widehat{U}_q(\mathfrak{sl}_2)$ is topologically isomorphic to the direct product of a countable family of full matrix algebras.

(A)$\Rightarrow$(B) since every finite-dimensional representation of $U_q(\mathfrak{sl}_2)$ is completely reducible.

Every factor corresponds to a finite-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$.

Outline of the proof. Analytic part(simple)

Lemma

$a, c \in$ Banach algebra. If a is invertible and $aca^{-1} = \gamma c$ for $\gamma \in \mathbb{C}$ with $|\gamma| \neq 1$ and $\gamma \neq 0 \Rightarrow c$ is nilpotent.

Proof.

Let $c^n \neq 0\ \forall n \in \mathbb{N}$.
$ac^n a^{-1} = \gamma^n c^n \Rightarrow \|\gamma^n c^n\| \leqslant \|a\|\,\|c^n\|\,\|a^{-1}\| \Rightarrow |\gamma|^n \leqslant \|a\|\,\|a^{-1}\|$
$\Rightarrow |\gamma| \leqslant 1$. Letting $d := aca^{-1}$ we have similarly $|\gamma|^{-1} \leqslant 1$.
Contradiction. □

Let $E, F, K \in B$ (Banach algebra) with quantum $\mathfrak{sl}_2$ relations. $KEK^{-1} = q^2 E$ and $KFK^{-1} = q^{-2} F$ that E and F are nilpotent. (The only analytic idea here.)

Algebraic part of the proof (formulas)

We does need to assume that B is Banach.
It is essential that q is not a root of unity and E and F are nilpotent.
Then $\forall m\ \exists$ a non-trivial Laurent polynomial in K that belongs to the ideal of $U_q(\mathfrak{sl}_2)$ generated by E^m.
Calculations $\Rightarrow$ the subalgebra generated by E, F, K is f.d.

Complex-analytic form

Putting $K := e^{\hbar H}$ and $\quad q := e^{\hbar} \Rightarrow \exists$ homomorphism

$$U_q(\mathfrak{sl}_2) \to \widetilde{U}(\mathfrak{sl}_2)_\hbar \colon E \to E, \quad F \to F, \quad K \to e^{\hbar H}.$$

Some questions (not all) on $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ can be reduced to the well-known representation theory of $U_q(\mathfrak{sl}_2)$.

Theorem (A., 2020)

$e^\hbar$ is not a root of unity (and $\sinh\hbar \neq 0$).
(A) The range of any cont. homomorphism from $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ to a Banach algebra is finite dimensional.
(B) Every continuous finite-dimensional representation of $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ is completely reducible.
(C) $\widetilde{U}(\mathfrak{sl}_2)_\hbar$ is topologically isomorphic to the direct product of a countable family of full matrix algebras.

Each f.d. irr.rep. of $U(\mathfrak{sl}_2)$ corresponds to two f.d. irr.reps. of $U_q(\mathfrak{sl}_2)$ and an infinite series of f.d. irr.reps.of $\widetilde{U}(\mathfrak{sl}_2)_\hbar$!

$U_q(\mathfrak{g})$ for an arbitrary semisimple complex Lie algebra $\mathfrak{g}$

By definition, the Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$ is generated by a tuple of triples E_j, F_j, K_j subject to relations similar to that for $\mathfrak{sl}_2$ (inside triples) + additional relations between triples.

Main results on $U_q(\mathfrak{sl}_2)$ also hold for an arbitrary $\mathfrak{g}$ when $|q| \neq 1$.

Theorem (A., 2020)

(A) The range of any homomorphism from $U_q(\mathfrak{g})$ to a Banach algebra is finite dimensional.
(B) The Arens-Michael envelope of $U_q(\mathfrak{g})$ is a product of full matrix algebras.

$\widetilde{U}(\mathfrak{g})_\hbar$ is also can be defined!

Table: Banach-space representations of $U_q(\mathfrak{g})$ with $\mathfrak{g}$ semisimple

q	**representations**	**irr. representations**
$\lvert q\rvert \neq 1$	f.d. compl. reducible	f.d.
not root, $\lvert q\rvert = 1$	no restriction known	$\exists$ top. irr. inf. d.
root	$\exists$ inf. d.	f.d. bounded degree

Table: Banach-space representations of $\widetilde{U}(\mathfrak{sl}_2)_\hbar$

$\mathbf{e}^\hbar$	**representations**	**irr. representations**
not root	f.d. compl. reducible	f.d.
root	$\exists$ inf .d	f.d. bounded degree

A piece of NC geometry (over $\mathbb{C}$)

Table: Spaces vs Algebras

Spaces	**Comm. algebras**	**NC**
locally compact	comm. C^*-algebras	C^*-algebras
affine varieties	comm. f.g. algebras	f.g. algebras
complex-analytic	comm. HFG algebras	HFG algebras

1) Gelfand-Naimark
2) AG
3) Forster + Pirkovskii (for f.d. Stein spaces)

Table: Groups vs Hopf algebras

Groups	**Comm. Hopf algs**	**NC**
locally compact	comm. C^*-algebraic HA	C^*-algebraic HA
affine algebraic	comm. f.g. HA	f.g. HA
complex Lie	?	?

2) Cartier's theorem: Every affine algebraic group scheme over a field of characteristic 0 is non-singular.
My suggestion is to use Hopf HFG algebras as analogues of f.d. Stein groups (at least, as first approximation)

Commutative Hopf HFG algebras.

Functor:

complex Lie groups $\to$ comm. Hopf HFG algebras: $G \mapsto \mathcal{O}(G)$

Standard operations on $\mathcal{O}(G)$ (function algebra):

$$\Delta(f)(g,h) = f(gh), \quad \varepsilon(f) = f(e), \quad S(f)(g) = f(g^{-1}),$$

Theorem (A., 2020)

The restriction of the functor to Stein groups is an anti-equivalence.

The argument is similar to the proof of Cartier's theorem

Outline of the proof:

Sufficiency: G is a Lie group $\Rightarrow \mathcal{O}(G)$ is a Stein algebra $\Rightarrow$ HFG.

Necessity.

Step 1. Forster+Pirkovskii: If H is a comm. Hopf HFG algebra $\Rightarrow$ $\exists$ a Stein space X s.t. $\mathcal{O}(X) \cong H$. Also X is a group.

Step 2. X is reduced. First we show that $\mathcal{O}_{X,e}$ is a Hopf $\widehat{\otimes}$-algebra. Second, as in Cartier's proof we show that the associated graded algebra of $\mathcal{O}_{X,e}$ is a polynomial algebra. No nilpotent elements.

Step 3. If there is a singular point, then all points are singular (X is a group), which is impossible.

Hence X is a manifold and so it is a Stein group.

So we can treat a general Hopf HFG algebra as a non-comm. analogue of the function algebra on a complex Lie group (at least Stein group).

Dual category — "complex-analytic quantum groups" or "quantum complex Lie groups".

Cocommutative Hopf HFG algebras.

For a Lie group: $\mathscr{A}(G) := \mathcal{O}(G)'$ (strong dual).
The Hopf algebra structure on $\mathbb{C}G$ (the group algebra):
$\Delta(\delta_g) = \delta_g \otimes \delta_g$, $\varepsilon(\delta_g) = 1$, $S(\delta_g) = \delta_{g^{-1}}$.
Extends to $\mathscr{A}(G)$.
Consider the AM envelope.

Theorem (A.)

G is compactly generated $\Leftrightarrow$ $\widehat{\mathscr{A}}(G)$ is a HFG algebra $\Leftrightarrow$ $\widehat{\mathscr{A}}(G)$ is a Fréchet algebra.

Outline of the proof. Compactly generated $\Rightarrow$ HFG (heavily based on the theory of LCSs). Three steps.

Step 1: G is simply connected. $\widehat{U}(\mathfrak{g}) \cong \widehat{\mathscr{A}}(G)$.
Step 2: G is connected. If $\widetilde{G}$ is the universal covering, then $\mathscr{A}(\widetilde{G}) \to \mathscr{A}(G)$ is a quotient map (this follows from Pták's open mapping theorem).

Step 3: G is compactly generated. G_0 is the connected component.
$\exists$ discrete group Γ and a quotient map $\mathbb{C}\Gamma * \mathscr{A}(G_0) \to \mathscr{A}(G)$ is (free product of $\widehat{\otimes}$-algebras).

Fréchet $\Rightarrow$ compactly generated: first for discrete, next the general case.

HFG $\Rightarrow$ Fréchet. Trivial.

Remark

The class of cocommutative Hopf HFG algebras is wider. If $\mathfrak{f}_n$ is the free Lie algebra in n generators, then $\widehat{U}(\mathfrak{f}_n) \cong \mathscr{F}_n$ is a Hopf HFG algebra but there no Lie group s.t. $\widehat{U}(\mathfrak{f}_n) \cong \widehat{\mathscr{A}(G)}$.

Structure of $\widehat{\mathscr{A}}(G)$.

Let G be connected and linear ($=\exists$ a faithful holomorphic finite-dimensional representation). First, look on the LCS structure. $\exists$ composition series for G:

$$E \subset B \subset G,$$

where G/B is linearly reductive, B/E and E are simply connected nilpotent.

Theorem (A., 2020)

As a LCS,

$$\widehat{\mathscr{A}}(G) \cong [U(\mathfrak{e})] \mathbin{\widehat{\otimes}} \widehat{\mathscr{A}}(B/E) \mathbin{\widehat{\otimes}} \widehat{\mathscr{A}}(G/B),$$

where $\mathfrak{e}$ is the Lie algebra of E and $[U(\mathfrak{e})]$ is the algebra of all formal power series w.r.t. PBW basis.

For linearly reductive: $\widehat{\mathscr{A}}(G/B)$ is a product of full matrix algebras.
For simply connected nilpotent: $\widehat{\mathscr{A}}(B/E)$ is a power series space (with restrictions on growth).
Look on the algebraic structure.

$$G = ((\cdots(F_1 \rtimes F_2) \rtimes \cdots) \rtimes F_{n-1}) \rtimes G/B,$$

where $F_j \cong \mathbb{C}$.

Theorem (A., 2022)

$$\widehat{\mathscr{A}}(G) \cong (\cdots(\mathbb{C}[[x_1]] \mathbin{\widehat{\#}} \cdots \mathbb{C}[[x_p]]) \mathbin{\widehat{\#}} \mathfrak{A}_{m-1}) \mathbin{\widehat{\#}} \cdots \mathfrak{A}_{m-1}) \mathbin{\widehat{\#}} \cdots \cdots \mathbin{\widehat{\#}} \mathfrak{A}_0) \mathbin{\widehat{\#}} \cdots \mathfrak{A}_0) \mathbin{\widehat{\#}} \widehat{\mathscr{A}}(G/B),$$

Here $\widehat{\#}$ is the sign of the analytic smash product, $\mathfrak{A}_0, \ldots, \mathfrak{A}_{m-1}$ is a certain algebra of power series with $\mathfrak{A}_0 = \mathcal{O}(\mathbb{C})$, p is the dimension of E.

Holomorphic duality

C^*-algebraic theory uses invariant weights as a basis for duality.

Complex-analytic approach admits a duality scheme without invariant weights.

Akbarov's holomorphic reflexivity diagram (for complex Lie group):

$$
\begin{array}{ccc}
\mathcal{O}(G) & \xleftarrow{\;AM\ env\;} & \mathcal{O}_{exp}(G) \\
\downarrow{\scriptstyle str.dual} & & \uparrow{\scriptstyle str.dual} \\
\mathscr{A}(G) & \xrightarrow[\;AM\ env\;]{} & \widehat{\mathscr{A}}(G)
\end{array}
$$

Vertical arrows are strong duals.
All spaces are Hopf $\widehat{\otimes}$-algebras.
$\mathcal{O}(G)$ and $\widehat{\mathscr{A}}(G)$ are Hopf HFG.
$\mathcal{O}(G)$ is called holomorphically reflexive if the diagram commutes.

Theorem (A.)

Let G be a complex Lie group with finitely many connected components and G_0 the component of 1. Then $\mathcal{O}(G)$ is holomorphically reflexive $\Leftrightarrow$ G_0 is linear.

History. Akbarov introduced this scheme (2008) and claimed optimistically that the reflexivity holds for G with countably many components and G_0 algebraic and conjectured (later) that G_0 can be assumed linear.
There was a gap in the proof and counterexamples for the countable case.

Conjecture: The restriction on components can be removed if we use a generalized linearity, i.e., $\exists$ faithful (possibly ∞-dim.) hol. representation on a Banach space.

Summary

Table: Groups vs Hopf algebras

Groups	Comm. Hopf algs	NC
locally compact	comm. C^*-algebraic HA	C^*-algebraic HA
affine algebraic	comm. f.g. HA	f.g. HA
Stein	comm. HFG HA	HFG HA

Non-trivial examples: $\widetilde{U}(\mathfrak{a}\mathfrak{f}_1)_\hbar$, $\widetilde{U}(\mathfrak{g})_\hbar$, $\widehat{U}_q(\mathfrak{g})$. (Life is better when $\hbar$ is a number not a letter!)

The commutative case is understood.

Some results in cocommutative case.

Duality theory without invariant weights.

Open problems and questions.

How does all this relate to C^*-algebraic quantum groups?

Find the structure of $\widetilde{U}(\mathfrak{g})_\hbar$ when $\mathfrak{g} \neq \mathfrak{sl}_2$.

Do exist infinite-dimensional irreducible Banach space representations of $U_q(\mathfrak{sl}_2)$ if $|q| = 1$ and q is not a root of unity?

Topologically irreducible i.-d. reps exist (see above).

Are the homomorphisms $\iota\colon U_q(\mathfrak{g}) \to \widehat{U}_q(\mathfrak{g})$ and $\theta\colon U_q(\mathfrak{g}) \to \widetilde{U}(\mathfrak{g})_\hbar$ always injective?

ι is injective when $|q| = 1$ (follows Pedchenko's result).

Find the structure of a general (not only associated with a Lie group) cocommutative Hopf HFG algebra.

Are $\widehat{U}_q(\mathfrak{g})$ and $\widetilde{U}(\mathfrak{g})_\hbar$ holomorphically reflexive?

References:

O. Yu. Aristov, *Holomorphically finitely generated Hopf algebras and quantum Lie groups*, arXiv:2006.12175.

O. Yu. Aristov, *Banach space representations of Drinfeld-Jimbo algebras and their complex-analytic forms*, Illinois J. Math. (to appear), arXiv:2012.12565.

O. Yu. Aristov, *The relation "Commutator Equals Function" in Banach algebras*, Math. Notes, 109:3 (2021), 323–334, arXiv:1911.03293.

O. Yu. Aristov, *On holomorphic reflexivity conditions for complex Lie groups*, Proc. Edinburgh Math. Soc. (2), 64:4 (2021), 800–821, arXiv: 2002.03617.

O. Yu. Aristov, *Decomposition of the algebra of analytic functionals on a connected complex Lie group and its completions into iterated analytic smash products*, arXiv:2209.04192 (in Russian).

Many thanks for your attention!